2nd International Joint Conference on Environmental and Light Industry Technologies,
21 – 22 November 2011, Budapest, Hungary
Óbuda University

# A Calculating Method for Approximative Determination of the Flow Rate of Small Rivers

Szabó, László

Óbuda University, RKK, Environmental Protection Engineering Institute

***Abstract:***

*The concept of this calculating method is based on the assumption that the distribution of the speeds along the river depth can be approximated by parabolas that slip on a very thin boundary layer on the bottom of the river. The thickness of this boundary layer,*

$\delta = O(1/\sqrt{\mathrm{Re}})$*, is negligible regarding the measured values of depth in different locations. The discrete points determined by data from the measurements in locations along the width of the river are connected by continuous and smooth lines that are generated by spline-interpolations. The calculation of transported quantity is done by integration between these lines.*

*The calculating method presented here is just a theoretical one. For its use to real data it is necessary to develope an adequate computer programme either in MATLAB or in Maple or in other suitable programming language. The accuracy of measurements highly influences the exactness of the calculation.*

***Keywords:*** *flow rate, approximation by parabolas, spline-interpolation*

## 1 INTRODUCTION

The importance of the determination of the flow of a smaller river has several ecological and other reasons. There are some procedures for the approximative calculation of the flow since the times before using of computers as treated in [1] and in [2]. An other one of the several calculating methods is based on calculation of the average area of transects and on that of middle surface speed and applying the formula Q=ALC/T, where A is the average area of two transects in a distance L from each other, C is a correctional coefficient depending on the type of the bottom and varying between 0.8 and 0.9, and T is the average time the water flows the distance L. T is to be measured several times in order to get a more accurate average value.

## 2 EXPERIMENTAL RESULTS AND DISCUSSION

The method to be presented here is based on few measured data acquired in different locations along the width of the river and on an experimental coefficient which depends on the type and quality of the bottom surface of the river bed.

The flow of a river depends on many circumstances as wether the river is tidal or ebbing or steady, because the water surface is respectively slightly convex or slightly concave or horizontal. The measurments have to take place on a straight stretch. Assuming the continuity of the flow the following equation holds:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \qquad (1)$$

and even so that every member in (1) is to be equal to zero, where $u,v,w$ are the components of the velocity of the flow.

Despite the straightness of the stretch the flow near the bottom is turbulent and it has a very thin boundary layer with the bottom of the river. The thickness of the boundary layer can be estimated as $\delta = O(\frac{1}{\sqrt{\mathrm{Re}}})$. Within the boundary layer the flow is laminar. The statements and calculations regarding the boundary layer with the bottom are detailed in [3] and [4].

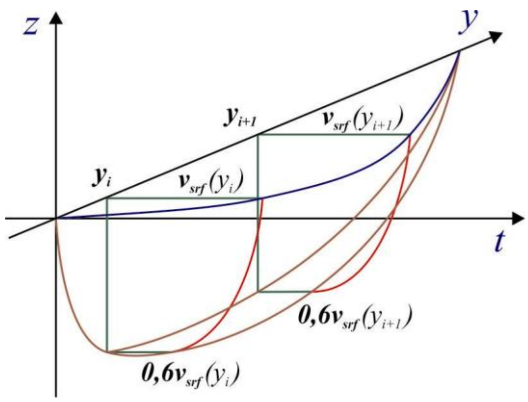


Figure 1: The distribution of speeds along the width and the depth of the river

The thickness is very small, therefore its value is negligible from the values of depth measured along the width of the river. Despite the turbulent phenomena near the bottom we assume that the thread elements of the water are straight in time $\Delta t$, that is, the lengths of their way can be regarded as values of a function of two variables.

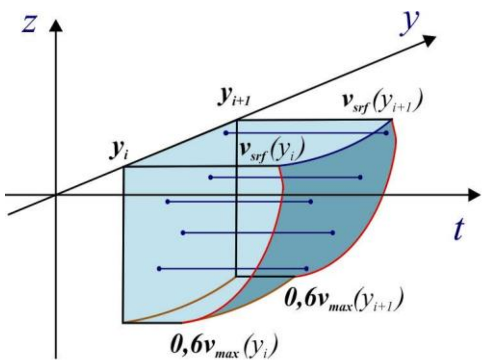


*Figure 2. Continuous and smooth distribution of speeds*

*between two neighbouring location*

The data to be measured are: the velocities of the flow on the surface: $v_{srf}(y_i)$, the values of the depth along the width in different locations, $b(y_i)$, $y_i \in [0, w]$, $i = 1,2,...,n$; $w$ denotes the width of the river.

As it is known the maximal velocity of the flow of a river is not necessarily on the surface, but at some deepness, so in case of a deeper river the values $v_{\max}(y_i)$ and their depths $z_m(y_i)$ are also to be measured, if even not entirely along the whole width, but at least in some locations in the main stream. The locations of the measurments are to be distributed uniformly, but not necessarily equidistant, along the width of the river. For the correctness of the dimensions we take the values of velocities as if multiplied by some $\Delta t$.

The concept of the calculation is based on the assumption, that the distribution of the speeds of the flow along the depth can be approximated by parabolas that slip on the boundary layer. Another new aspect of this calculating method is the feature that the points determined by discrete measured values are connected by continuous and smooth lines generated by spline-interpolations. The integration is done between these lines in order to get the quantity of water transported by the river in the time $\Delta t$, [T].

Let us denote $v_{srf}(y) = v_{srf}$, $v_{\max}(y) = v_{\max}$, [L/T], $b(y) = b$, $z_m(y) = z_m$, [L].

Two cases will be discerned according to the location of the vertex of the parabolas.

Case 1: the vertex of the parabola of the distribution of speeds along the depth is on the surface of the river which means that $v_{srf} = v_{\max}$.

Now we determine the coefficients of a general equation of second degree,

$$f(z) = a_1 z^2 + a_2 z + a_3, \quad a_1 \neq 0.$$

In this case $f(z)$ is an even function, so

$f(z) = f(-z) = a_1 z^2 - a_2 z + a_3$ implies $a_2 = 0$.

$f(z) = a_1 z^2 + a_3$, for $z = 0$ we obtain $f(0) = a_3 = v_{srf}$.

For $z = b$ we get $f(b) = a_1 b^2 + v_{srf}$.

The experiments indicate that the speed of flow on the bottom is $0.6 v_{\max}$. This experimental coefficient can vary in value depending on the type and quality of the surface of bottom. In a case of further simplification it can also be taken as $0.6 v_{srf}$.

$a_1 b^2 + v_{srf} = 0.6 v_{srf}$ thus we get $a_1 = -\dfrac{0.4 v_{srf}}{b^2}$.

If we have got measured data for both the speeds on the surface and the depths in $n-1$ inner locations along the width of the river, then the quantity Q, $[\mathrm{L}^3]$, can be calculated by integration. Both the surface speed and the depth are zero at the both banks.

$$Q=\sum_{i=0}^{n-1}\int_{y_i}^{y_{i+1}}\int_{0}^{b}\left(\frac{-0.4v_{srf}}{b^2}z^2+v_{srf}\right)dzdy=$$

$$=\sum_{i=0}^{n-1}\int_{y_i}^{y_{i+1}}\left[\frac{-0.4v_{srf}}{b^2}\frac{z^3}{3}+v_{srf}z\right]_0^b dy=$$

$$=\sum_{i=0}^{n-1}\int_{y_i}^{y_{i+1}}b\frac{-0.4v_{srf}+3v_{srf}}{3}dy=$$

$$=\frac{13}{15}\sum_{i=0}^{n-1}\int_{y_i}^{y_{i+1}}b(y)v_{srf}(y)dy. \qquad (2)$$

Between two measured values $v_{srf}(y_i)$ and $v_{srf}(y_{i+1})$, respectively $b(y_i)$ and $b(y_{i+1})$, $v_{srf}(y)$ and $b(y)$ as functions of $y\in[0,w]$ are to be interpolated by a set of polynomials of degree 3. This set of plynomials constitutes a spline-interpolation of their curves which are continous and smooth, that is, they are differentiable with respect to $y$.

Now we briefly review the method and procedure of the execution of a spline-interpolation by polynomials of degree 3. A description of splines is to be found in [5].

Let n+1 data $f_i = f(y_i)$ be given of an $f(y)$ function, $i = 0,1,\ldots$n.

Let $p=(p_0, p_1,....p_{n-1})$ be a set of polynomials, each of them of degree 3.

For the polynomials let the follows hold:

$$p_i(y_i)=p_{i-1}(y_i)$$

$$p_i'(y_i)=p_{i-1}'(y_i)$$

$$p_i''(y_i)=p_{i-1}''(y_i) \qquad i=1,2,...,n-1 \qquad (3)$$

$$p_0''(y_0)=p_{n-1}''(y_n)=0 \qquad (4)$$

These equalities mean that the polynomial $p$ is continuous and smooth in every $y_i$ location, furthermore that here the measure of the curvature of $p$ equals to those of the both adjacent polynomials.

Let $p$ interpolate function $f(y)$, that is, $p(y_i)=p_i(y_i)=f(y_i)=f_i$ and

$p(y_{i+1})=p_{i+1}(y_{i+1})=p_i(y_{i+1})=f(y_{i+1})=f_{i+1}$.

Let us introduce $s_i = p'(y_i)$, that is, $p_i'(y_i)=s_i$ and $p_i'(y_{i+1})=s_{i+1}$.

Let $t$ and $h_i$ be the following: $t = \dfrac{y - y_i}{h_i}$ and $h_i = y_{i+1} - y_i$, furthermore $\dfrac{dt}{dy} = h_i^{-1}$ and $\left(\dfrac{dt}{dy}\right)^2 = h_i^{-2}$.

Let us introduce a new polynomial for $p_i(y)$, of which coefficients we will get from the Hermite-interpolation for the two endpoints: $q_i(t) = p_i(y) = p_i(y_i + h_i t)$, where $t \in [0,1]$.

Thus, $q_i(0) = p_i(y_i) = f_i$ and $q_i(1) = p_i(y_{i+1}) = f_{i+1}$ and

$$p_i''(y) = \frac{d^2 p_i(y)}{dy^2} = \frac{d^2 q_i(t)}{dt^2} \frac{dt^2}{dy^2} = q_i''(t) h_i^{-2}.$$

From the known formula of the Hermite-interpolation we get:

$$q_i(t) = (1-t)^2 f_i + t^2 f_{i+1} + t(1-t)((1-t)(2f_i + h_i s_i) + t(2f_{i+1} - h_i s_{i+1})).$$

Differentiating $q_i$ twice we get:

$$q_i''(0) = 6(f_{i+1} - f_i) - 2h_i(2s_i + s_{i+1}) \quad \text{and}$$

$$q_i''(1) = -6(f_{i+1} - f_i) + 2h_i(s_i + 2s_{i+1}).$$

Our next task is to determine the values of $s_i$, $i = 0,1,2,\ldots,\mathrm{n}$.

From (4) we get:

$$p_0''(y_0) = q_0''(0) h_0^{-2} = 3h_0^{-2}(f_1 - f_0) - h_0^{-1}(2s_0 + s_1) = 0, \text{ that is,}$$

$$h_0^{-1}(2s_0 + s_1) = 3h_0^{-2}(f_1 - f_0).$$

From (3) we get:

$$q_i''(0) h_i^{-2} = q_{i-1}''(1) h_{i-1}^{-2}, \text{ that is,}$$

$$3h_i^{-2}(f_{i+1} - f_i) - h_i^{-1}(2s_i + s_{i+1}) = -3h_{i-1}^{-2}(f_{i+1} - f_i) + h_{i-1}^{-1}(s_i + 2s_{i+1}), \text{ that is,}$$

$$h_{i-1}^{-1}(s_i + 2s_{i+1}) + h_i^{-1}(2s_i + s_{i+1}) = 3h_{i-1}^{-2}(f_{i+1} - f_i) + 3h_i^{-2}(f_{i+1} - f_i), \quad i = 1,2,\ldots,\mathrm{n}-1$$

From (4) we get again:

$p''_{n-1}(y_n) = q''_{n-1}(1)h_0^{-2} = -3h_{n-1}^{-2}(f_n - f_{n-1}) - h_{n-1}^{-1}(2s_{n-1} + s_n) = 0$, that is,

$h_{n-1}^{-1}(2s_{n-1} + s_n) = 3h_{n-1}^{-2}(f_n - f_{n-1})$.

This constitutes a system of linear equation of n+1 equations for $s_i$, $i = 0,1,2,...,n$.

$$\mathbf{A}=\begin{bmatrix} \frac{2}{h_0} & \frac{1}{h_1} & 0 & & & & & 0 \\ \frac{1}{h_0} & \frac{2}{h_0}+\frac{2}{h_1} & \frac{1}{h_1} & 0 & & & & 0 \\ 0 & \frac{1}{h_1} & \frac{2}{h_1}+\frac{2}{h_2} & \frac{1}{h_2} & 0 & & & 0 \\ 0 & 0 & \frac{1}{h_2} & \frac{2}{h_2}+\frac{2}{h_3} & \frac{1}{h_3} & 0 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & \frac{1}{h_{n-3}} & \frac{2}{h_{n-3}}+\frac{2}{h_{n-2}} & \frac{1}{h_{n-2}} & 0 \\ 0 & & & & 0 & \frac{1}{h_{n-2}} & \frac{2}{h_{n-2}}+\frac{2}{h_{n-1}} & \frac{1}{h_{n-1}} \\ 0 & 0 & 0 & & 0 & 0 & \frac{1}{h_{n-1}} & \frac{2}{h_{n-1}} \end{bmatrix}$$

$$\mathbf{s}=\begin{bmatrix} s_0 & s_1 & s_2 & \dots & \dots & s_{n-2} & s_{n-1} & s_n \end{bmatrix}$$

$$\mathbf{d}=\begin{bmatrix} \frac{3(f_1-f_0)}{h_0^{\,2}} & \frac{3(f_1-f_0)}{h_0^{\,2}}+\frac{3(f_2-f_1)}{h_1^{\,2}} & \dots & \dots & \frac{3(f_{n-1}-f_{n-2})}{h^2{}_{n-2}}+\frac{3(f_n-f_{n-1})}{h^2{}_{n-1}} & \frac{3(f_n-f_{n-1})}{h^2{}_{n-1}} \end{bmatrix}$$

$$As^T = d^T \qquad (5)$$

The solution of (5) provides the values of **s**.

The system has a unique solution for **s**, because for every **s**≠0 $sAs^T > 0$, so matrix A is positive definite.

In case 1 the procedure above is to be executed twice, first the $f_i$ -s take the values of $v_{srf}(y_i)$, thereafter they take those of $b(y_i)$ .

Let $q_i^{\ b}$ and $q_i^{\ v_{srf}}$ denote the polynomials obtained by the spline-interpolations, then (2) takes the form:

$$Q = \frac{13}{15} \sum_{i=0}^{n-1} h_i \int_0^1 q_i^{\ b}(t) q_i^{\ v_{srf}}(t) dt \,. \qquad (6)$$

Case 2:

Now let us examine the case when the vertex of the approximating parabola is under the surface of the river, at the depth of $z_m(y_i)$. In this case the calculation needs two further types of data, namely the values of $v_{\max}(y_i)$ and those of $z_m(y_i)$. A parabola of which directrix is parallel to z-axis, can be determined uniquely by its three points. Only one restriction is to be observed for the points, namely none of them is allowed to be the vertex of the parabola. It is because none of the three given points will necessary be the vertex of the unique parabola running through them.

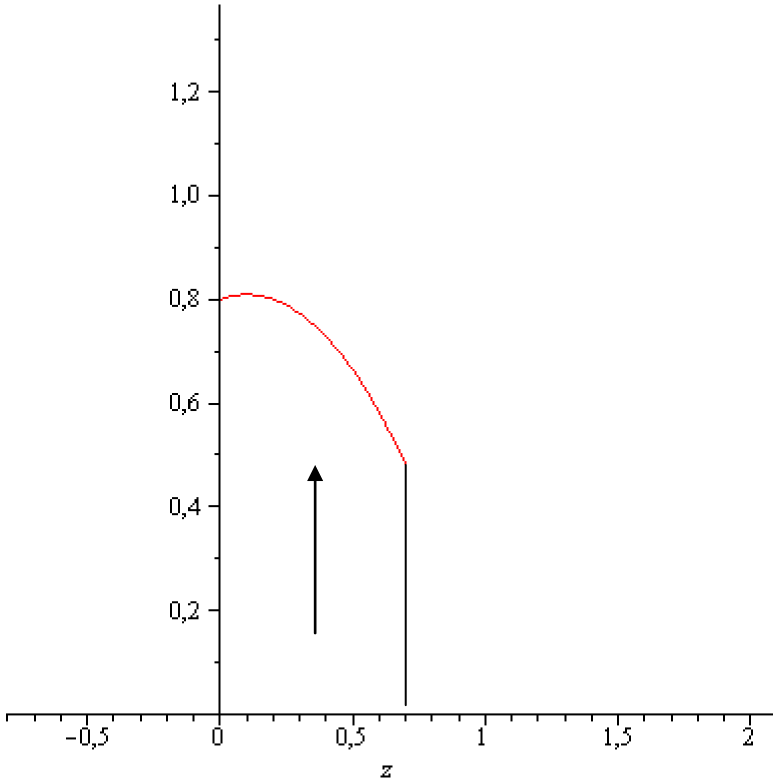


*Figure 3. In case 2 the vertex is below of the surface*

The general form of the equation of such a parabola is: $\begin{vmatrix} z^2 & z & f(z) & 1 \\ z_1{}^2 & z_1 & f(z_1) & 1 \\ z_2{}^2 & z_2 & f(z_2) & 1 \\ z_3{}^2 & z_3 & f(z_3) & 1 \end{vmatrix} = 0$

The known points are:

$$(z_1, f(z_1)) = (0, v_{srf}),\ (z_2, f(z_2)) = (2z_m, v_{srf}),\ (z_3, f(z_3)) = (b, 0.6v_{\max})$$

The point $(z_m, v_{\max})$ is expected to be vertex of the parabola, but it approximates the point only.

$$\begin{vmatrix} z^2 & z & f(z) & 1 \\ 0 & 0 & v_{srf} & 1 \\ 4z_m{}^2 & 2z_m & v_{srf} & 1 \\ b^2 & b & 0.6v_{\max} & 1 \end{vmatrix} = 0$$

To obtain the equation in the form $f(z) = c_1 z^2 + c_2 z + c_3$ we expand the above determinant in two steps.

$$z^2 \begin{vmatrix} 0 & v_{srf} & 1 \\ 2z_m & v_{srf} & 1 \\ b & 0.6v_{\max} & 1 \end{vmatrix} - z \begin{vmatrix} 0 & v_{srf} & 1 \\ 4z_m{}^2 & v_{srf} & 1 \\ b^2 & 0.6v_{\max} & 1 \end{vmatrix} + f(z) \begin{vmatrix} 0 & 0 & 1 \\ 4z_m{}^2 & 2z_m & 1 \\ b^2 & b & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & v_{srf} \\ 4z_m{}^2 & 2z_m & v_{srf} \\ b^2 & b & 0.6v_{\max} \end{vmatrix} = 0$$

After expanding the above determinants and after simplifications we get:

$$f(z) = \frac{0.6v_{\max} - v_{srf}}{b(b - 2z_m)} z^2 + \frac{2z_m v_{srf} - 1.2 z_m v_{\max}}{b(b - 2z_m)} z + v_{srf} \qquad (7)$$

For (7), as an other type of parabolas, the integration and thereafter the necessary spline-interpolations shall be done.

In practice the calculating $Q$ may require the integration of the both types of parabolas, because the measurments of data $v_{\max}(y_i)$ and $z_m(y_i)$ seem to be necessary in locations of the main stream and eventually in those of deeper place only. In such a case, let us say, three measured data mean five data, because the data set includes the initial and final zero data.

In order to reduce the number of measurements further simplifications can be done.

Substituting $v_{\max}$ by $v_{srf}$, that is, at the bottom, on the boundary layer the speed of flow is substituted by $0.6v_{srf}$, then from (7) we get:

$$f(z) = \frac{-0.4v_{srf}}{b(b - 2z_m)} z^2 + \frac{0.8 z_m v_{srf}}{b(b - 2z_m)} z + v_{srf} \qquad (8)$$

If we suppose that $z_m(y) = cb(y)$ and we guess the value of $c$ as an other experimental coefficient, thus, the number of measurements will be reduced to the necessary minimum. So the double integration of (8) needs again polynomials $q_i^{\,b}$ and $q_i^{\,v_{srf}}$ only.

## 3 CONCLUSION

This calculating method can be put into practice by developing programmes in an adequate language for the measured data.

For real data a system of programmes to be developed could prove the practical usability and some closer accuracy of this calculating method.

### References

[1] Bogdánfy Ö.: *HIDRAULIKA*, BUDAPEST, 1904., PP. 220 – 39, PP. 403 – 417.
[2] Kendi Finály I.: *HIDROTECHNIKA I-II-III*., BUDAPEST, 1926., VOL. II. PP. 70 – 78.
[3] Prandtl, L.: *STRÖMUNGSLEHRE,* BRAUNSCHWEIG, 1942., PP. 110 – 120.
[4] Hughes, T.J.R., Marsden, J. E.: *A SHORT COURSE IN FLUID MECHANICS*, BERKELY, 1976. ,PP. 126 – 140.
[5] Henrici, P.: *ESSENTIALS OF NUMERICAL ANALYSIS WITH POCKET CALCULATOR DEMONSTRATIONS,* NEW YORK, 1982, (ALSO IN HUNGARIAN: *NUMERIKUS ANALÍZIS*, BUDAPEST, 1986., PP. 236 – 249.)